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Cascading RG flows from new Sasaki-Einstein manifolds

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Abstract: In important recent developments, new Sasaki-Einstein spaces $Y^{p,q}$ and conformal gauge theories dual to $AdS_5 \times Y^{p,q}$ have been constructed. We consider a stack of N D3-branes and M wrapped D5-branes at the apex of a cone over $Y^{p,q}$. Replacing the D-branes by their fluxes, we construct asymptotic solutions for all p and q in the form of warped products of the cone and $R^{3,1}$. We show that they describe cascading RG flows where N decreases logarithmically with the scale. The warp factor, which we determine explicitly, is a function of the radius of the cone and one of the coordinates on $Y^{p,q}$. We describe the RG cascades in the dual quiver gauge theories, and find an exact agreement between the supergravity and the field theory β-functions. We also discuss certain dibaryon operators and their dual wrapped D3-branes in the conformal case $M = 0$.

Keywords: Duality in Gauge Field Theories, AdS-CFT and dS-CFT Correspondence, Supergravity Models.

Contents

1. Introduction

An interesting generalization of the basic AdS/CFT correspondence [1]–[3] results from studying branes at conical singularities [4]–[8]. Consider a stack of N D3-branes placed at the apex of a Ricci-flat 6-d cone Y_6 whose base is a 5-d Einstein manifold X_5. Comparing the metric with the D-brane description leads one to conjecture that type IIB string theory on $AdS_5 \times X_5$ with N units of 5-form flux, is dual to the low-energy limit of the world volume theory on the D3-branes at the singularity.

Well-known examples of X_5 are the orbifolds $\mathbf{S}^5/\Gamma$ where Γ is a discrete subgroup of SO(6) [4]. In these cases X_5 has the local geometry of a 5-sphere. Constructions of the dual gauge theories for Einstein manifolds X_5 which are not locally equivalent to $\mathbf{S}^5$ are also possible. The simplest example is $X_5 = T^{1,1} = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\,\mathrm{U}(1)$ [6]. The dual gauge theory is the conformal limit of the world volume theory on a stack of N D3-branes placed at the apex of the conifold [6, 7], which is a cone over $T^{1,1}$. This $\mathcal{N}=1$ superconformal gauge theory is $\mathrm{SU}(N)\times\mathrm{SU}(N)$ with bifundamental fields A_i, B_j, $i,j=1,2$, and a quartic superpotential.

Recently, a new infinite class of Sasaki-Einstein manifolds $Y^{p,q}$ of topology $S^2\times S^3$ was discovered [9, 10]. Following progress in [11], the $\mathcal{N}=1$ superconformal gauge theories dual to $AdS_5\times Y^{p,q}$ were ingeniously constructed in [12]. These quiver theories have gauge groups $\mathrm{SU}(N)^{2p}$, bifundamental matter, and marginal superpotentials involving both cubic and quartic terms. These constructions generalize the SCFT on D3-branes placed at the apex of the complex cone over dP_1 [13], corresponding to $Y^{2,1}$ [11]. Impressive comparisons of the conformal anomaly coefficients between the AdS and the CFT sides were carried out for dP_1 in [14], and in full generality in [12].

In this paper we address a number of further issues concerning the gauge/gravity duality involving the $Y^{p,q}$ spaces. We match the spectra of dibaryon operators in the gauge theory with that of wrapped D3-branes in the string theory. Next, we consider gauge theories that arise upon addition of M wrapped D5-branes at the apex of the cone. Our discussion generalizes that given in [15, 16] for the $Y^{2,1}$ case. We show that these gauge theories can undergo duality cascades, and construct the dual warped supergravity solutions with $(2,1)$ flux.[1]

As a preliminary, in the next two sections we review the gauge theory duals for and the geometry of these $Y^{p,q}$ spaces.

2. The conformal surface of $Y^{p,q}$ gauge theories

In this section, we review the construction of the $Y^{p,q}$ gauge theories and argue that they flow to an IR conformal "fixed surface" of dimension two. That this surface has dimension two will be more or less clear from the gravity side where the two free complex parameters are $C - ie^{-\phi}$ and $\int_{S^2}(C_2 - ie^{-\phi}B_2)$.

As derived in [12], the quivers for these $Y^{p,q}$ gauge theories can be constructed from two basic units, σ and τ. These units are shown in figure 1.

To construct a general quiver for $Y^{p,q}$, we define some basic operations with σ and τ. First, there are the inverted unit cells, $\tilde{\sigma}$ and $\tilde{\tau}$, which are mirror images of σ and τ through a horizontal plane. To glue the cells together, we identify the double arrows corresponding to the U^α fields on two unit cells. The arrows have to be pointing in the same direction for the identification to work. So for instance we may form the quiver $\sigma\tilde{\tau} = \tilde{\tau}\sigma$, but $\sigma\tau$ is not allowed. In this notation, the first unit cell is to be glued not only to the cell on the right but also to the last cell in the chain. A general quiver might look like

$$\sigma\tilde{\sigma}\sigma\tilde{\tau}\tau\tilde{\sigma}\,. \tag{2.1}$$

In general, a $Y^{p,q}$ quiver consists of p unit cells of which q are of type σ. The $Y^{p,p-1}$ gauge theories will have only one τ type unit cell, while the $Y^{p,1}$ theories will have only one σ type unit cell.

Each node of the quiver corresponds to a gauge group while each arrow is a chiral field transforming in a bifundamental representation. For the $Y^{p,q}$ spaces, there are four types of bifundamentals labeled U^α, V^α, Y, and Z where $\alpha = 1$ or 2. To get a conformal theory, we take all the gauge groups to be $\mathrm{SU}(N)$. Later in this paper, when we add D5-branes, we will change the ranks of some of the gauge groups and break the conformal symmetry.

[1] The duality cascade was first developed for the conifold in [17, 18] and later generalized in [19, 15, 16].

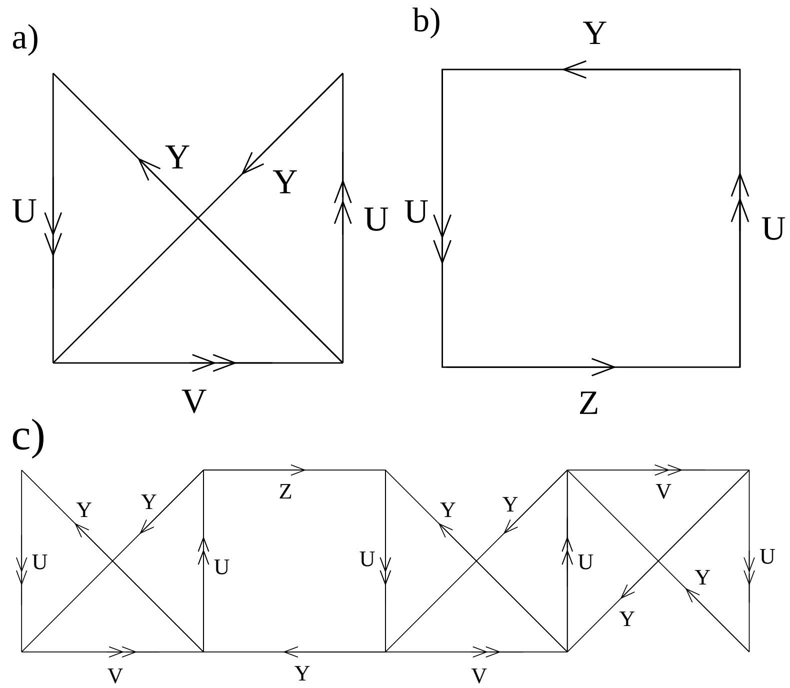

Figure 1: Shown are *a*) the unit cell σ; *b*) the unit cell τ; and *c*) the quiver for $Y^{4,3}$, $\sigma\tilde{\tau}\sigma\tilde{\sigma}$.

The superpotential for this quiver theory is constructed by summing over gauge invariant operators cubic and quartic in the fields U^α, V^α, Y, and Z. For each σ unit cell in the gauge theory, we add two cubic terms to the superpotential of the form

$$\epsilon_{\alpha\beta} U_L^\alpha V^\beta Y \qquad \text{and} \qquad \epsilon_{\alpha\beta} U_R^\alpha V^\beta Y \,. \tag{2.2}$$

Here, the indices R and L specify which group of U^α enter in the superpotential, the U^α on the right side or the left side of σ. The trace over the color indices has been suppressed. For each τ unit cell, we add the quartic term

$$\epsilon_{\alpha\beta} Z U_R^\alpha Y U_L^\beta \,. \tag{2.3}$$

An analysis of the locus of conformal field theories begins with counting the fundamental degrees of freedom which are in this case the $2p$ gauge couplings and the $p+q$ superpotential couplings (assuming an unbroken SU(2) symmetry for the U^α and V^α). We will assume all the gauge groups have equal ranks. There are in total $3p+q$ fields and thus $3p+q$ anomalous dimensions which we can tune to get a conformal theory. We think of the $3p+q$ β-functions as functions of the $3p+q$ anomalous dimensions which are in turn functions of the $3p+q$ coupling strengths, $\beta_j(\gamma_i(g_k))$.

Let us check that one set of solutions of $\beta_j = 0$ involves setting the anomalous dimensions of all the Z fields equal, the anomalous dimensions of all the Y fields equal, and similarly for the U^α and V^α. Instead of working with the anomalous dimensions γ, of the fields, we find it convenient to work with the R charges, R_Y, R_Z, R_U, and R_V. (For superconformal gauge theories, recall that $2(1+\gamma) = 3R$.)

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There are $p+q$ β-functions for the superpotential couplings. $p-q$ of the β functions vanish when $R_Z + R_Y + 2R_U = 2$ and are associated with loops in the τ unit cells, while the remaining $2q$ vanish when $R_U + R_Y + R_V = 2$ and are associated to loops in the σ unit cells.

There are $2p$ β-functions for the gauge couplings. $2q$ of these couplings are associated with the σ unit cells, and the beta functions for these couplings vanish when $2 = R_U + R_V + R_Y$ while the remaining $2p-2q$ belong to the τ unit cells and vanish when $R_Z + R_Y + 2R_U = 2$. Thus the gauge coupling β-functions contain exactly the same information as the superpotential β-functions.

It could be that there are more solutions to setting the $\beta_j = 0$ which involve more generic values for the anomalous dimensions. However, such solutions would require even more degeneracy among the $3p+q$ β-functions, which is unlikely. Assuming we have found the most general solution of $\beta_j = 0$ (which we have checked for dP_1 but should be checked in general), we have found that only $3p+q-4+2$ of the $\beta_j = 0$ are linearly independent. Thus there is seemingly a two dimensional plane in the space of allowed anomalous dimensions which produce conformal field theories. Of course we know that a-maximization [20] will pick out the right anomalous dimensions.

However, there is a different way of looking at these $3p+q-2$ linearly independent β-functions. They place $3p+q-2$ constraints on the $3p+q$ couplings, leaving a space of conformal theories with two complex dimensions. By construction, this space preserves the $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$ global flavor symmetry of the $Y^{p,q}$. If we allow a breaking of this symmetry, then there may exist additional exactly marginal superpotential deformations (see [21]).

3. Review of the $Y^{p,q}$ geometry

The $Y^{p,q}$ spaces are topologically $S^2 \times S^3$, and the Sasaki-Einstein metric on them takes the form [9, 10]

$$\begin{aligned} d\Omega^2_{Y^{p,q}} &= \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)v(y)}dy^2 + \frac{v(y)}{9}(d\psi - \cos\theta d\phi)^2 + \\ &\quad + w(y)[d\alpha + f(y)(d\psi - \cos\theta d\phi)]^2 \end{aligned} \tag{3.1}$$

where

$$w(y) = \frac{2(b-y^2)}{1-y}, \tag{3.2}$$

$$v(y) = \frac{b-3y^2+2y^3}{b-y^2}, \tag{3.3}$$

$$f(y) = \frac{b-2y+y^2}{6(b-y^2)}. \tag{3.4}$$

For the metric to be complete,

$$b = \frac{1}{2} - \frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}. \tag{3.5}$$

The coordinate y is allowed to range between the two smaller roots of the cubic $b-3y^2+2y^3$:

$$y_1 = \frac{1}{4p}\left(2p-3q-\sqrt{4p^2-3q^2}\right), \tag{3.6}$$

$$y_2 = \frac{1}{4p}\left(2p+3q-\sqrt{4p^2-3q^2}\right). \tag{3.7}$$

The three roots of the cubic satisfy $y_1+y_2+y_3=3/2$, so the biggest root, which we will need later in the paper, is

$$y_3 = \frac{1}{4p}\left(2p+2\sqrt{4p^2-3q^2}\right). \tag{3.8}$$

The period of α is $2\pi\ell$ where

$$\ell = -\frac{q}{4p^2 y_1 y_2}. \tag{3.9}$$

The remaining coordinates are allowed the following ranges: $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, and $0 \leq \psi < 2\pi$.

The volume of $Y^{p,q}$ is given by

$$\mathrm{Vol}(Y^{p,q}) = \frac{q(2p+\sqrt{4p^2-3q^2})\ell\pi^3}{3p^2}. \tag{3.10}$$

4. Dibaryons and new 3-cycles

We will identify some new supersymmetric 3-cycles in the $Y^{p,q}$ geometry, but first recall that Martelli and Sparks [11] identified two supersymmetric 3-cycles, denoted Σ_1 and Σ_2 in their paper. These three cycles are obtained by setting $y=y_1$ or $y=y_2$ respectively. At these values for y, the circle parametrized by ψ shrinks to zero size, and the three cycles can be thought of as a U(1) bundle parametrized by α over the round S^2 parametrized by θ and ϕ.

Martelli and Sparks [11] computed the R-charges of the dibaryons corresponding to D3-branes wrapped on Σ_1 and Σ_2. In general, these R-charges are given by the formula [22]

$$R(\Sigma_i) = \frac{\pi N}{3}\frac{\mathrm{Vol}(\Sigma_i)}{\mathrm{Vol}(Y^{p,q})}. \tag{4.1}$$

From this general formula, it follows that

$$R(\Sigma_1) = \frac{N}{3q^2}\left(-4p^2+2pq+3q^2+(2p-q)\sqrt{4p^2-3q^2}\right), \tag{4.2}$$

$$R(\Sigma_2) = \frac{N}{3q^2}\left(-4p^2-2pq+3q^2+(2p+q)\sqrt{4p^2-3q^2}\right). \tag{4.3}$$

These R-charges should correspond to operators $\det(Y)$ and det(Z) made out of the bifundamental fields that are singlet under the global SU(2) symmetry. Dividing these dibaryon R-charges by N, we observe a perfect match with the R-charges of the Y and Z singlet fields determined from gauge theory by Benvenuti, Franco, Hanany, Martelli, and Sparks [12], $R_Y = R(\Sigma_1)/N$ and $R_Z = R(\Sigma_2)/N$.[2]

[2] The gauge theory computation for $Y^{2,1}$ was performed earlier by [14].

Here we show which 3-cycles correspond to the dibaryons made out of the SU(2) doublet fields U^α and V^α.[3] Such dibaryons carry spin $N/2$ under the global SU(2). On the string side, the wrapped D3-brane should therefore have an SU(2) collective coordinate (see [22] for an analogous discussion in the case of $T^{1,1}$). The only possibility is that this SU(2) is precisely the SU(2) of the round S^2 in the metric. Therefore, the 3-cycles corresponding to these dibaryons should be localized at a point on the S^2.

Now recall from the gauge theory analysis of [12] that

$$R_U = \frac{(2p(2p-\sqrt{4p^2-3q^2}))}{3q^2}, \tag{4.4}$$

$$R_V = \frac{(3q-2p+\sqrt{4p^2-3q^2})}{3q}. \tag{4.5}$$

Before proceeding, note that $R_V = R_U + R_Z$. So if we determine which cycle Σ_3 corresponds to U^α, we can deduce that V^α is just a sum of Σ_3 and Σ_2.

As discussed above, the three cycle Σ_3 should correspond to fixing a point on the S^2 and integrating over the fiber. Setting $\phi = \theta = \text{const}$, the induced metric on this three cycle becomes

$$ds^2 = \frac{1}{wv}dy^2 + \frac{v}{9}d\psi^2 + w(d\alpha + f d\psi)^2. \tag{4.6}$$

We can characterize this 3-cycle more precisely. The metric on Σ_3 can be thought of as a principal U(1) bundle over an S^2 where the S^2 is parametrized by y and ψ. A principal U(1) bundle over S^2 is a Lens space $S^3/\mathbb{Z}_k$ where k is given by the first Chern class c_1 of the fibration. The $A = f d\psi$ is a connection one-form on the U(1) bundle. Because α ranges from 0 to $2\pi\ell$, $dA = 2\pi c_1/\ell$. Integrating c_1 over the S^2 yields

$$\int_{S^2} c_1 = \frac{f(y_2) - f(y_1)}{\ell} = -p. \tag{4.7}$$

In other words, our Σ_3 is the Lens space $S^3/\mathbb{Z}_p$. In [11], Σ_1 and Σ_2 were identified as the Lens spaces $S^3/\mathbb{Z}_{p+q}$ and $S^3/\mathbb{Z}_{p-q}$ respectively.

We find

$$\text{Vol}(\Sigma_3) = \int \sqrt{g}\, dy\, d\alpha\, d\psi = \frac{4\pi^2\ell}{3}(y_2 - y_1), \tag{4.8}$$

where we have used the fact from (4.6) that $\sqrt{g} = 1/3$. Plugging into the formula for the R-charge, indeed $R(\Sigma_3) = N R_U$.

We now imagine that V^α corresponds to adding the cycles Σ_3 and Σ_2 together. Indeed, these two cycles intersect along a circle at $y = y_2$.

We also check that Σ_3 is a supersymmetric cycle, or in other words that the form $\frac{1}{2}J \wedge J$, where J is the Kähler form on the cone over $Y^{p,q}$, restricts to the induced volume form on the cone over Σ_3. More formally, we are checking that Σ_3 is calibrated by $\frac{1}{2}J \wedge J$.

[3] We would like to thank S. Benvenuti and J. Sparks for discussions about these new 3-cycles. A similar analysis will likely appear in a revision of [12].

From Martelli and Sparks [11, (2.24)], we find that

$$J = r^2 \frac{1-y}{6} \sin\theta \, d\theta \wedge d\phi + \\ + \frac{1}{3} r \, dr \wedge (d\psi - \cos\theta \, d\phi) - d(yr^2) \wedge \left(d\alpha + \frac{1}{6}(d\psi - \cos\theta \, d\phi) \right),$$

and hence

$$J|_{\Sigma_3} = \frac{1}{3} r dr \wedge d\psi + d(yr^2) \wedge \left(d\alpha + \frac{1}{6} d\psi \right). \tag{4.9}$$

Thus we find that

$$\frac{1}{2} J \wedge J \bigg|_{\Sigma_3} = \frac{r^3}{3} dr \wedge d\psi \wedge dy \wedge d\alpha \tag{4.10}$$

as expected.

5. Warped solutions with (2,1) flux

The first step in constructing supersymmetric warped solutions for these $Y^{p,q}$ spaces is constructing a harmonic $(2,1)$ form $\Omega_{2,1}$. We begin by rewriting the metric so that locally we have a U(1) fiber over a Kähler-Einstein manifold. From [11, (2.17)], we have

$$d\Omega^2_{Y^{p,q}} = (e^\theta)^2 + (e^\phi)^2 + (e^y)^2 + (e^\beta)^2 + (e^\psi)^2 \tag{5.1}$$

where we have defined the one forms

$$e^\theta = \sqrt{\frac{1-y}{6}} d\theta \,, \qquad e^\phi = \sqrt{\frac{1-y}{6}} \sin\theta d\phi \,, \tag{5.2}$$

$$e^y = \frac{1}{\sqrt{wv}} dy \,, \qquad e^\beta = \frac{\sqrt{wv}}{6} (d\beta + \cos\theta d\phi) \,, \tag{5.3}$$

$$e^\psi = \frac{1}{3} (d\psi - \cos\theta d\phi + y(d\beta + \cos\theta d\phi)) \,. \tag{5.4}$$

In terms of the original coordinates $\beta = -6\alpha - \psi$. Here, the ψ is a coordinate on the local U(1) fiber.

There is then a local Kähler form, denoted J_4 by [11], on the Kähler-Einstein base:

$$J_4 = e^\theta \wedge e^\phi + e^y \wedge e^\beta \,. \tag{5.5}$$

Based on [15], we expect to be able to construct $\Omega_{2,1}$ from a $(1,1)$ form ω using this local Kähler-Einstein metric such that $*_4 \omega = -\omega$, $d\omega = 0$, and $\omega \wedge J_4 = 0$. We guess that

$$\omega = F(y)(e^\theta \wedge e^\phi - e^y \wedge e^\beta) \,. \tag{5.6}$$

The form ω is clearly anti-selfdual and orthogonal to J_4. Using a complex basis of one-forms constructed in [11, (2.27)], it is not hard to check that ω is indeed a $(1,1)$ form. The condition $d\omega = 0$ then implies that

$$F(y) = \frac{1}{(1-y)^2} \,. \tag{5.7}$$

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Further, we construct a $(2,1)$ form from the wedge product of a $(1,0)$ form and ω:

$$\Omega_{2,1} = K\left(\frac{dr}{r} + ie^{\psi}\right) \wedge \omega\,. \tag{5.8}$$

We have introduced a normalization constant K for later convenience. We have checked that $d\Omega_{2,1} = 0$ and $*_6\Omega_{2,1} = i\Omega_{2,1}$.

Next, we analyze

$$\int_{\Sigma_i} \Omega_{2,1} \tag{5.9}$$

for the three three-cycles $i = 1,\,2,\,3$. We find that

$$\int_{\Sigma_1} \Omega_{2,1} = -K\frac{8i\pi^2\ell}{3}\frac{y_1}{1-y_1}\,, \tag{5.10}$$

$$\int_{\Sigma_2} \Omega_{2,1} = -K\frac{8i\pi^2\ell}{3}\frac{y_2}{1-y_2}\,, \tag{5.11}$$

$$\int_{\Sigma_3} \Omega_{2,1} = -K\frac{4i\pi^2\ell}{3}\left(\frac{1}{1-y_2} - \frac{1}{1-y_1}\right). \tag{5.12}$$

The ratios between these integrals look superficially to be irrational. However, the ratios must be rational, and we find that if we set

$$K = \frac{9}{8\pi^2}(p^2 - q^2) \tag{5.13}$$

then

$$\int_{\Sigma_1} \Omega_{2,1} = -i(-p+q)\,, \tag{5.14}$$

$$\int_{\Sigma_2} \Omega_{2,1} = -i(p+q)\,, \tag{5.15}$$

$$\int_{\Sigma_3} \Omega_{2,1} = -ip\,. \tag{5.16}$$

Now, to construct a supergravity solution, we take the real RR F_3 and NSNS H_3 forms to be

$$iK'\Omega_{2,1} = F_3 + \frac{i}{g_s}H_3\,, \tag{5.17}$$

$$F_3 = -KK'e^{\psi} \wedge \omega\,, \qquad H_3 = g_sKK'\frac{dr}{r} \wedge \omega\,, \tag{5.18}$$

where we have introduced another normalization constant K'. In particular, F_3 should be quantized such that

$$\int_{\Sigma_1} F_3 = 4\pi^2\alpha' M(p-q)\,, \tag{5.19}$$

where M is the number of D5-branes. Thus we find that $K' = 4\pi^2\alpha' M$. (See [23] for our normalization conventions.)

5.1 Derivation of five-form flux

For the metric and F_5 we take the usual ansatz with the warp factor h,

$$ds^2 = h^{-1/2}dx_4^2 + h^{1/2}(dr^2 + r^2 d\Omega^2_{Y^{p,q}})\,, \tag{5.20}$$

$$g_s F_5 = d(h^{-1}) \wedge d^4x + *[d(h^{-1}) \wedge d^4x]\,. \tag{5.21}$$

Due to the appearance of the y-dependent factor $F(y)$ in the $(2,1)$ flux, it is inconsistent to assume that h is a function of r only. Instead, similar to the gravity duals of fractional branes on the $\mathbb{Z}_2$ orbifold [24], h is a function of two variables, r and y. For $q \ll p$ the y-dependence can be ignored, and the warp factor approaches that found for the warped conifold in [17]. On the other hand, for $p - q \ll p$ we find that h gets sharply peaked near $y = 1$, and the solutions approach the gravity duals of fractional branes in orbifold theories [25, 26, 24]. Thus, the warped solutions we find with the $Y^{p,q}$ serve as interesting interpolations between the conifold and the orbifold cases.

More explicitly, the first term in (5.21) is

$$-h^{-2}\left(\frac{\partial h}{\partial r}dr + \sqrt{wv}\frac{\partial h}{\partial y}e^y\right) \wedge d^4x\,. \tag{5.22}$$

Working out its Hodge dual, and substituting into the equation

$$dF_5 = H_3 \wedge F_3\,, \tag{5.23}$$

we find the second order PDE

$$-(1-y)\frac{\partial}{\partial r}\left(r^5\frac{\partial h}{\partial r}\right) - r^3\frac{\partial}{\partial y}\left((1-y)wv\frac{\partial h}{\partial y}\right) = \frac{C}{r(1-y)^3}\,, \tag{5.24}$$

where $C \equiv 2(g_s K K')^2$. Note that, after dividing the PDE by $r^5(1-y)$, we obtain the standard equation

$$-\nabla^2_{pq} h = \frac{1}{6}|H_3|^2\,, \tag{5.25}$$

where ∇^2_{pq} is the laplacian on the cone over $Y^{p,q}$.

Below we display a system of first-order equations which any supersymmetric solution has to obey. Our supergravity solution, based on $\Omega_{2,1}$ is expected to be supersymmetric if it has no curvature singularities [27]. Naively, this first order system could be easier to solve than the second order PDE (5.24). Such a first order system for F_5 can be generated starting from the ansatz

$$F_5 = B_2 \wedge F_3 + dC_4\,, \tag{5.26}$$

where

$$g_s C_4 = h(r,y)^{-1}d^4x + \frac{f(r,y)}{(1-y)\sqrt{wv}}e^\psi \wedge e^\theta \wedge e^\phi \wedge e^\beta\,, \tag{5.27}$$

and we have used (6.2). Enforcing the selfdual constraint $F_5 = *F_5$, one finds

$$\frac{\partial h}{\partial r}r^5 = \frac{\partial f}{\partial y}\frac{1}{1-y} + \frac{C}{(1-y)^4}\ln r\,,$$

$$\frac{\partial h}{\partial y}r^3(1-y)wv = -\frac{\partial f}{\partial r}\,,$$

which is indeed a first order system (a similar type of system appears in a somewhat different context in [28]). Unfortunately, as it involves one more function than our PDE (5.24), it seems no easier to solve; in fact this system is equivalent to (5.24) as a constraint on $h(r, y)$.

5.2 Solving for the warp factor

First, we discuss the boundary conditions at $y = y_1$ and $y = y_2$. At these points the radius of the circular coordinate ψ smoothly shrinks to zero. Defining the coordinate $\rho \sim \sqrt{y - y_1}$ near the boundary, we find that the metric in these two dimensions (with other coordinates fixed) is locally

$$ds_2^2 = d\rho^2 + \rho^2 d\psi^2 \,. \tag{5.28}$$

The behavior of ψ-independent modes in these radial coordinates is well-known. The boundary condition is $\frac{dh}{d\rho} = 0$, so that

$$h = h_0 + h_2\rho^2 + \cdots = h_0 + \tilde{h}_2(y - y_1) + \cdots \,. \tag{5.29}$$

In terms of the y-coordinate, we have the boundary conditions that $\frac{\partial h}{\partial y}$ is finite at the boundaries, while h is positive there.

Let us substitute into (5.24)

$$h = r^{-4} f(t, y)\,, \qquad t = \ln\left(\frac{r}{r_0}\right) . \tag{5.30}$$

The PDE for $f(t, y)$ assumes the simpler form

$$(1-y)\left(-\frac{\partial^2 f}{\partial t^2} + 4\frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial y}\left(2(b - 3y^2 + 2y^3)\frac{\partial f}{\partial y}\right) = \frac{C}{(1-y)^3}\,. \tag{5.31}$$

Now, it is clear that there are solutions of the form

$$f(t, y) = At + s(y)\,, \tag{5.32}$$

where A is a constant, and the ODE for $s(y)$ is

$$-\frac{d}{dy}\left(2(b - 3y^2 + 2y^3)\frac{ds}{dy}\right) = \frac{C}{(1-y)^3} - 4A(1-y)\,. \tag{5.33}$$

The boundary conditions are that s' is finite at both end-points. Therefore, integrating the LHS from y_1 to y_2 we must find zero. This imposes a constraint on A that

$$\int_{y_1}^{y_2} dy \left[\frac{C}{(1-y)^3} - 4A(1-y)\right] = 0\,, \tag{5.34}$$

whose solution is

$$A = \frac{C}{4(1-y_1)^2(1-y_2)^2}\,. \tag{5.35}$$

Now we can integrate (5.33) twice to find

$$s(y) = -\frac{C}{4(b-1)}\left[\frac{1}{1-y} + \frac{(1+2y_1)(1+2y_2)\ln(y_3 - y)}{2(b-1)}\right] + \text{const}\,. \tag{5.36}$$

This function has singularities at $y = y_3$ and $y = 1$, but they are safely outside the region $y_1 < y < y_2$ for all admissible p and q. To summarize, the warp factor we find is

$$h(r, y) = \frac{A \ln(r/r_0) + s(y)}{r^4} \,. \tag{5.37}$$

Just like the solution found in [17], this solution has a naked singularity for small enough r. It should be interpreted as the asymptotic form of the solution. In the conifold case, the complete solution [18] involves the deformation of the conifold that is important in the IR, but in the UV the solution indeed approaches the asymptotic form found earlier in [17]. Finding the complete solutions for cones over $Y^{p,q}$, non-singular in the IR, remains an important problem.

There are two interesting special limits of our solutions. For $q \ll p$,

$$y_1 = -\frac{3q}{4p} + O\left(\frac{q^2}{p^2}\right), \qquad y_2 = \frac{3q}{4p} + O\left(\frac{q^2}{p^2}\right). \tag{5.38}$$

In this limit the range of y becomes narrow, and both end-points approach zero. Since $\frac{\partial h}{\partial y}$ is finite, the variation of h in the y-direction can be ignored, and we have $h \sim \ln(r/r_0)/r^4$, as in [17]. This is not surprising, since for $q \ll p$ the spaces $Y^{p,q}$ may be approximated by a $\mathbb{Z}_p$ orbifold of $T^{1,1}$.

The other special case is $q = p - l$, with $l \ll p$. Now

$$b = 1 - \frac{27l^2}{4p^2} + O\left(\frac{l^3}{p^3}\right), \tag{5.39}$$

and

$$y_1 = -\frac{1}{2} + \frac{3l^2}{2p^2} + O\left(\frac{l^3}{p^3}\right), \qquad y_2 = 1 - \frac{3l}{2p} + O\left(\frac{l^2}{p^2}\right), \qquad y_3 = 1 + \frac{3l}{2p} + O\left(\frac{l^2}{p^2}\right). \tag{5.40}$$

Note that y_2 approaches 1 from below, while y_3 from above, as $\frac{l}{p} \to 0$. In this limit, we find that h depends on y strongly and gets sharply peaked near $y = 1$. While $\frac{\partial h}{\partial y}$ is finite at y_2 for any finite l and p, it diverges in the limit $l/p \to 0$. The limiting form of the warp factor is

$$h(r, y) \to 6\frac{(\alpha' g_s M)^2 p^4}{r^4}\left[\frac{4}{3}\ln\left(\frac{r}{r_0}\right) + \frac{1}{1-y} - \frac{2}{3}\ln(1-y)\right]. \tag{5.41}$$

To facilitate comparison with the solution found for the $S^5/\mathbb{Z}_2$ orbifold case in [24], it is convenient to introduce a new coordinate ρ

$$\frac{2}{3}(1-y) = 1 - \frac{\rho^2}{r^2} \,. \tag{5.42}$$

For $q = p$ the variable ρ ranges from from 0 to r. We also introduce an auxiliary radial variable $r' = \sqrt{r^2 - \rho^2}$.

The geometry of $Y^{p,p}$ is that of the $\mathbb{Z}_p$ orbifold of $S^5/\mathbb{Z}_2$. In [10], the space $Y^{1,1}$ was identified with the $\mathcal{N} = 2$ preserving $S^5/\mathbb{Z}_2$ orbifold. In the limit $q \to p$, the metric (3.1) is independent of both p and q. Only the period of the U(1) fiber coordinate α, which becomes π/p in this limit, depends on p. In the limit $p = q$, we can rewrite the metric on the cone over (3.1), $dr^2 + r^2 d\Omega^2_{Y^{p,p}}$, in the form

$$ds^2 = dr'^2 + \frac{1}{4}r'^2 \left[d\theta^2 + \sin^2\theta\, d\phi^2 + (-d\psi - 2d\alpha + \cos\theta\, d\phi)^2\right] + d\rho^2 + 4\rho^2(d\alpha)^2 \,. \quad (5.43)$$

From this form of the metric, one can see that the cone over $Y^{p,p}$ factors into a cone over an orbifolded S^3 and a cone over an orbifolded S^1. The cone over S^3 is locally $\mathbb{C}^2$ parametrized by θ, ϕ, $\psi + 2\alpha$, and the auxiliary radial coordinate r'. In this Euler angle parametrization, $-\psi - 2\alpha$ gives the overall phase of $(z_1, z_2) \in \mathbb{C}^2$. The cone over S^1 is parametrized by the angle α and the radial coordinate ρ.

The orbifold action sends $\alpha \to \alpha - \pi/p$, acting as $\mathbb{Z}_p$ on this cone over S^1. If the range of α ran from zero to π instead of from zero to π/p, then the cone over S^1 would be smooth. This same action shifts the phase $\psi + 2\alpha \to \psi + 2\alpha - 2\pi/p$. For the S^3 to be unorbifolded, the Euler angle $-\psi - 2\alpha$ should run from zero to 4π. We conclude the orbifold acts on this cone over S^3 as $\mathbb{Z}_{2p}$.

Putting the two cones together we find the $\mathbb{Z}_{2p}$ orbifold of $\mathbb{C}^3$ described in [12]. More precisely, the cone over $Y^{p,p}$ is the orbifold generated by $\zeta : (z_1, z_2, z_3) \to (\omega^{a_1} z_1, \omega^{a_2} z_2, \omega^{a_3} z_3)$, where ω is a $2p$'th root of unity and, keeping track of the signs of the angles, $\vec{a} = (1, 1, -2)$. We see that ζ^p generates a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_{2p}$. Moreover, ζ^p acts as the identity on z_3, fixing a circle in $Y^{p,p}$.

From this discussion, $y = 1$ (or equivalently $\rho = r$) is the location of the circle fixed by ζ^p. In terms of the coordinate ρ, our warp factor (5.41) becomes

$$h(r, \rho) \to 4\frac{(\alpha' g_s M)^2 p^4}{r^4}\left[\ln\left(\frac{r^4}{r^2 - \rho^2}\right) + \frac{r^2}{r^2 - \rho^2} + \text{const}\right]. \quad (5.44)$$

This matches the warp factor [24, (44)] exactly.

6. Matching the β function

In this section we match the supergravity and gauge theory calculations of the beta function. On the supergravity side, we can calculate the running of the gauge coupling constant g on the stack of D5-branes from the integral of B_2 (recall $dB_2 = H_3$) over a two-cycle. In particular

$$\frac{8\pi^2}{g^2} = \frac{1}{2\pi\alpha' g_s}\int_{\mathcal{C}} B_2 \,. \quad (6.1)$$

Now

$$B_2 = (\ln r)(4\pi^2\alpha' g_s M) K\omega \,. \quad (6.2)$$

It is unclear how to describe the two-cycle $\mathcal{C}$ in terms of the metric coordinates. However, based on [15], we expect that the harmonic form Poincare dual to $\mathcal{C}$ is $K\, e^\psi \wedge \omega$. Thus, we take

$$K\int_C \omega = K^2 \int_{Y^{p,q}} e^\psi \wedge \omega \wedge \omega \,. \quad (6.3)$$

One quickly finds

$$K\int_C \omega = \frac{p^2}{2\pi}\left(p + \sqrt{4p^2 - 3q^2}\right) \quad (6.4)$$

and hence that

$$\frac{8\pi^2}{g^2} = (\ln r) M p^2 \left(p + \sqrt{4p^2 - 3q^2}\right) . \qquad (6.5)$$

On the gauge theory side, we expect that

$$\beta_{D5} = \sum s^i \beta_i \qquad (6.6)$$

where the vector s^i describes how adding a D5-brane changes the ranks of the gauge groups. In [15], it was demonstrated that a cubic anomaly involving the R and $\mathrm{U}(1)_B$ charges is related in a precise way to this particular weighted sum of β functions:

$$\mathrm{tr} R\, \mathrm{U}(1)_B^2 = -\frac{2}{3M} \sum s^i \beta^i . \qquad (6.7)$$

In the derivation of this formula, it was assumed that the anomalous dimensions of the chiral fields are determined by the R-charges of the conformal theory. In principle, there could be coupling constant corrections to these anomalous dimensions if we start at a point away from the conformal surface described in section 2 and then add D5-branes. Even if we start on the conformal surface, the addition of D5-branes could conceivably introduce M/N corrections to these anomalous dimensions. The fact that the geometric and gauge theory calculations will agree indicates that these corrections should begin at order $(M/N)^2$, as discussed in [15].

Using R-charges for the chiral fields (4.2), (4.3), (4.4), and (4.5), which were first derived for $Y^{2,1}$ in [14] and later for all $Y^{p,q}$ in [12] using a-maximization, we can compute

$$\begin{aligned} \mathrm{tr} R\, \mathrm{U}(1)_B^2 &= (p-1)(R_Z - 1)(p+q)^2 + 2p(R_U - 1)(-p)^2 + \\ &\quad + 2q(R_V - 1)q^2 + (p+q)(R_Y - 1)(p-q)^2 \\ &= -\frac{2}{3} p^2 \left(p + \sqrt{4p^2 - 3q^2}\right) , \end{aligned}$$

which agrees with the intersection calculation above.

This calculation seems like a bit of magic. As part of a more general discussion of Seiberg duality [29] cascades, we will repeat this calculation using brute force for two classes $Y^{p,p-1}$ and $Y^{p,1}$ of spaces.

7. Cascades in the dual gauge theories

The simplest example of a Seiberg duality cascade occurs in the $\mathrm{SU}(N+M) \times \mathrm{SU}(N+2M)$ gauge theory with bifundamental fields A_i, B_j, $i, j = 1, 2$, and a quartic superpotential [17, 18]. (The theory with $M = 0$ is conformal — the addition of the M D5-branes breaks the conformal symmetry.) In this case the gauge coupling of $\mathrm{SU}(N + 2M)$ blows up after a finite amount of RG flow. To continue the flow beyond this point, one applies the duality transformation to this gauge group [18]. After this transformation, and an interchange of the two gauge groups, the new gauge theory is $\mathrm{SU}(\tilde{N} + M) \times \mathrm{SU}(\tilde{N} + 2M)$ with the same matter and superpotential, with $\tilde{N} = N - M$. This self-similar structure of the gauge theory under Seiberg duality is the crucial fact that allows the cascade to happen. If $N = kM$, where k is an integer, then the cascade stops after k steps, and we find $\mathrm{SU}(M) \times \mathrm{SU}(2M)$ gauge theory. This IR gauge theory exhibits a multitude of interesting effects visible in the dual supergravity background, such as confinement, and chiral symmetry breaking [18]. Particularly interesting is the apearance of an entire "baryonic branch" of the moduli space in the gauge theory [18, 30], whose existence in the dual supergravity was recently confirmed in [31, 32]. The presence of the baryonic operators in the IR gauge theory is related to the fact that for the $\mathrm{SU}(2M)$ gauge group, the number of flavors equals the number of colors.

The self-similar structure of the gauge theory under the duality, which allows the cascade to occur, can be found in more complicated quiver diagrams as well. In [15, 16] the cascade in the gauge theory dual to $AdS_5 \times Y^{2,1}$ was analyzed. The relevant gauge theory is $\mathrm{SU}(N+M)\times\mathrm{SU}(N+3M)\times\mathrm{SU}(N+2M)\times\mathrm{SU}(N+4M)$. If the initial conditions are such that the biggest gauge group flows to infinite coupling first, then after applying a duality transformation to this group and permuting factor groups, we find exactly the same theory, with $N \to N-M$. For a generic choice of initial conditions, the biggest gauge group will flow to infinite coupling again, and the cascade repeats until N reaches zero far in the infrared.

In fact, this structure of the cascade is possible for all gauge theories dual to $AdS_5 \times Y^{p,p-1}$ and $AdS_5 \times Y^{p,1}$.

7.1 Cascades for $Y^{p,p-1}$

As shown in [12], the systematics of the quiver diagram emerges most clearly for $p > 2$ where, placing the gauge groups at the vertices of a regular polygon, we find that the outer edge of the diagram consists of $2p$ vertices connected by double arrows pointing in the same direction, except for one "impurity" where the double arrow is replaced by a single one. The effect of the impurity is also to merge two inner single arrows into one (see figure 2). In the language of section 2, the $Y^{p,p-1}$ gauge theories consist of $(p-1)$ σ unit cells and one τ unit cell.

Upon addition of M fractional branes, the single arrow "impurity" connects the smallest gauge group $\mathrm{SU}(N+M)$ with the biggest gauge group $\mathrm{SU}(N+2pM)$. In the case of $p = 4$ corresponding to figure 2, the action of the Seiberg duality on $\mathrm{SU}(N+8M)$ gives $\mathrm{SU}(N)$ because the group effectively has $2N+8M$ flavors. Then we permute the adjacent vertices corresponding to $\mathrm{SU}(N)$ and $\mathrm{SU}(N+4M)$ to find a quiver identical to the one we started with, except with $N \to N-M$. Compared to the original diagram, the impurity moved two steps clockwise around the outer edge.

For the general p, there are $2p$ gauge groups. On the conformal surface, the gauge groups are all $\mathrm{SU}(N)$. However, we can add M D5-branes which shift the gauge groups to

$$\prod_{i=1}^{2p} \mathrm{SU}(N_i)\,, \tag{7.1}$$

where

$$N_{2n-1} = N + nM\,, \qquad N_{2n} = N + (p+n)M\,. \tag{7.2}$$

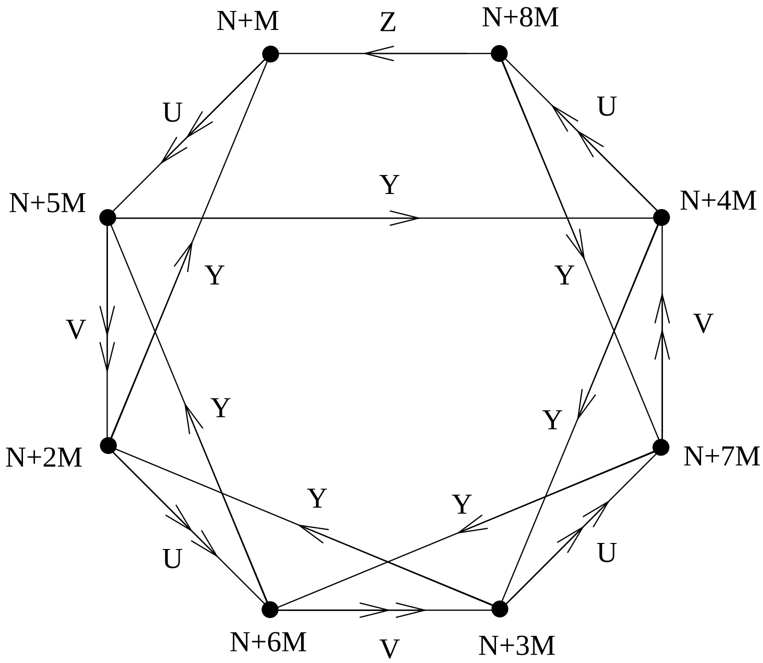

Figure 2: The quiver for $Y^{4,3}$ reproduced from [12, figure 4]. This quiver is identical to figure 1*c*.

To be painfully explicit, the gauge group becomes

$$\begin{aligned}&\mathrm{SU}(N+M)\times\mathrm{SU}(N+(p+1)M)\times\\&\times\mathrm{SU}(N+2M)\times\mathrm{SU}(N+(p+2)M)\times\cdots\\&\cdots\times\mathrm{SU}(N+pM)\times\mathrm{SU}(N+2pM)\,.\end{aligned}\tag{7.3}$$

Clearly, this action of Seiberg duality generalizes to higher p. The action on the biggest gauge group $\mathrm{SU}(N+2pM)$ reduces it to $\mathrm{SU}(N)$. Subsequent permutation of adjacent vertices $\mathrm{SU}(N)$ and $\mathrm{SU}(N+pM)$ turns the quiver into the one we started with, but with $N\to N-M$.

We now check that the gauge group with the most colors $\mathrm{SU}(N+2pM)$ is also the gauge group with the largest β function. All β functions are proportional to M. Setting $M=1$, we find

$$\beta_1 = 3+\frac{3}{2}\left[2(p+1)(R_U-1)+2(R_Y-1)+2p(R_Z-1)\right]\,,\tag{7.4}$$

$$\begin{aligned}\beta_{2n} &= 3(n+p)+\frac{3}{2}[2(n+1)(R_V-1)+(n+p+1)(R_Y-1)+\\&\qquad+(n+p-1)(R_Y-1)+2n(R_U-1)]\,,\end{aligned}\tag{7.5}$$

$$\begin{aligned}\beta_{2n-1} &= 3n+\frac{3}{2}[2(p+n)(R_U-1)+(n+1)(R_Y-1)+\\&\qquad+(n-1)(R_Y-1)+2(p+n-1)(R_V-1)]\,,\end{aligned}\tag{7.6}$$

$$\beta_{2p} = 6p+\frac{3}{2}\left[(R_Z-1)+(2p-1)(R_Y-1)+2p(R_U-1)\right]\,,\tag{7.7}$$

where for the β_{2n}, $1\le n\le p-1$ and for the β_{2n-1}, $2\le n\le p$.

Using the fact that the superpotential has R-charge two, we see that $R_U+R_V+R_Y=2$ and $2R_U+R_Y+R_Z=2$, from which it follows that

$$\begin{aligned}\beta_1 = -\beta_{2p} &= \frac{3}{2}(R_Y - R_Z - 2p(R_U+R_Y)) \,, \\ \beta_{2n-1} = -\beta_{2n} &= 3(1-R_V-pR_Y) \,.\end{aligned}$$

From (4.2), (4.3), (4.4), and (4.5), one can check that

$$\beta_1 < \beta_{2n-1} < 0 < \beta_{2n} < \beta_{2p} \,. \tag{7.8}$$

In particular,

$$\beta_1 = -5p + \sqrt{p^2-(p-1)^2} < 0 \,, \tag{7.9}$$

for $p \geq 1$. Moreover, consider the difference

$$\beta_{2n-1} - \beta_1 = \frac{2p^2\left(2p-\sqrt{p^2+6p-3}\right)}{(1-p)^2} \,. \tag{7.10}$$

This difference is strictly greater than zero for $p \geq 1$. We conclude that β_1 and β_{2p} have the largest magnitude of the $2p$ β-functions. Therefore, as the theory flows to the IR, the coupling will generically blow up first for the biggest gauge group $\mathrm{SU}(N+2pM)$, necessitating an application of Seiberg duality.

To make sure that we did not make a mistake, we check that

$$\beta_{D5} = \sum s^i \beta_i = \sum_{n=1}^{p} n\beta_{2n-1} + \sum_{n=1}^{p}(p+n)\beta_{2n} \,, \tag{7.11}$$

where the s^i is the D5-brane vector. Lo and behold,

$$\sum s^i \beta_i = p^2\left(p+\sqrt{4p^2-3(p-1)^2}\right) M \,, \tag{7.12}$$

in agreement with (6.5).

7.2 Cascades for $Y^{p,1}$

The $Y^{2,1}$ theory is not only the simplest example of $Y^{p,p-1}$ but also of $Y^{p,1}$. The $Y^{p,1}$ quivers, in the language of section 2, contain $(p-1)$ τ unit cells and one σ unit cell. The quiver for $Y^{4,1}$ is shown as figure 3.

The gauge groups for the $Y^{p,1}$ spaces are

$$\prod_{i=1}^{2p} \mathrm{SU}(N_i) \tag{7.13}$$

where

$$N_{2n-1} = N + (p+n)M \,; \qquad N_{2n} = N + nM \,, \tag{7.14}$$

where the σ unit cell contains both the first and second and also the last and second to last gauge groups.

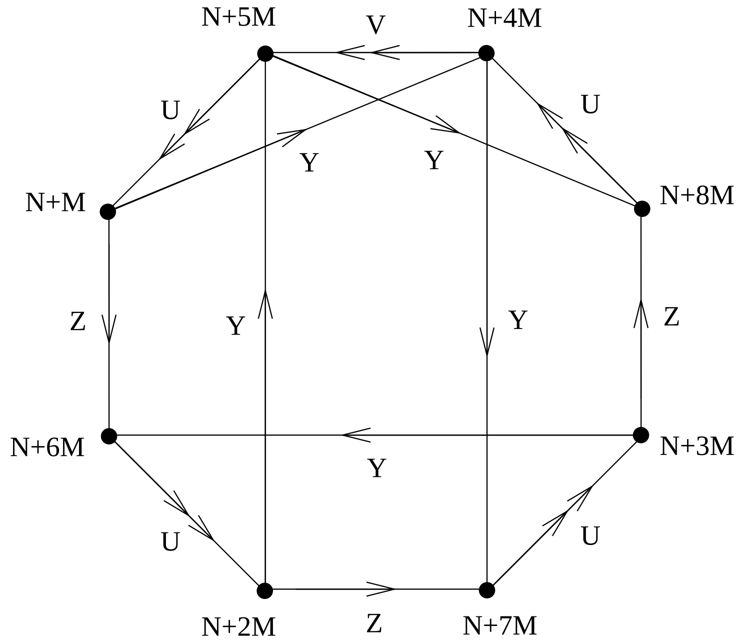

Figure 3: The quiver theory for $Y^{4,1}$, involving three τ unit cells and one σ unit cell.

The gauge groups with the largest and smallest numbers of colors are associated with the impurity, i.e. the σ unit cell. The gauge group with the largest number of colors $\mathrm{SU}(N + 2pM)$ has $2N + 2pM$ flavors. Thus, after a Seiberg duality, the gauge group will change to $\mathrm{SU}(N)$. Switching this $\mathrm{SU}(N)$ gauge group with its neighbor $\mathrm{SU}(N + pM)$ we find the same quiver but with the σ impurity shifted one cell to the left and $N \to N - M$ (see figure 4).

We now check whether Seiberg duality will generically happen at the gauge group with the largest number of colors. The β-functions for the $2p$ gauge groups are

$$\begin{aligned}\beta_1 = -\beta_{2p} &= 3(p - 1 + (1 - p)R_U + R_Y)M\,,\\ \beta_{2n+1} = -\beta_{2n} &= 3\left(p + \frac{1}{2}R_Y - \frac{1}{2}R_Z\right)M\,,\end{aligned}$$

where $1 \le n < p$. From (4.2), (4.3), (4.4), and (4.5), one can check that

$$\beta_{2n} < \beta_{2p} < 0 < \beta_1 < \beta_{2n+1}\,. \tag{7.15}$$

Indeed, the gauge group with the largest number of colors has the largest β function. However, an important difference between the $Y^{p,p-1}$ and the $Y^{p,1}$ gauge theories is that in the present case, there are $p-2$ other gauge groups which share the same large β-function. It may happen that Seiberg duality occurs first at the node with the largest number of colors, but the situation is less generic than before.

Finally, we check the sum

$$\sum_{i=1}^{2p} s^i \beta_i = \sum_{n=1}^{p}(p+n)\beta_{2n-1} + \sum_{n=1}^{p} n\beta_{2n} = p^2\left(p + \sqrt{4p^2-3}\right)M\,. \tag{7.16}$$

This result agrees with our expectations from (6.5).

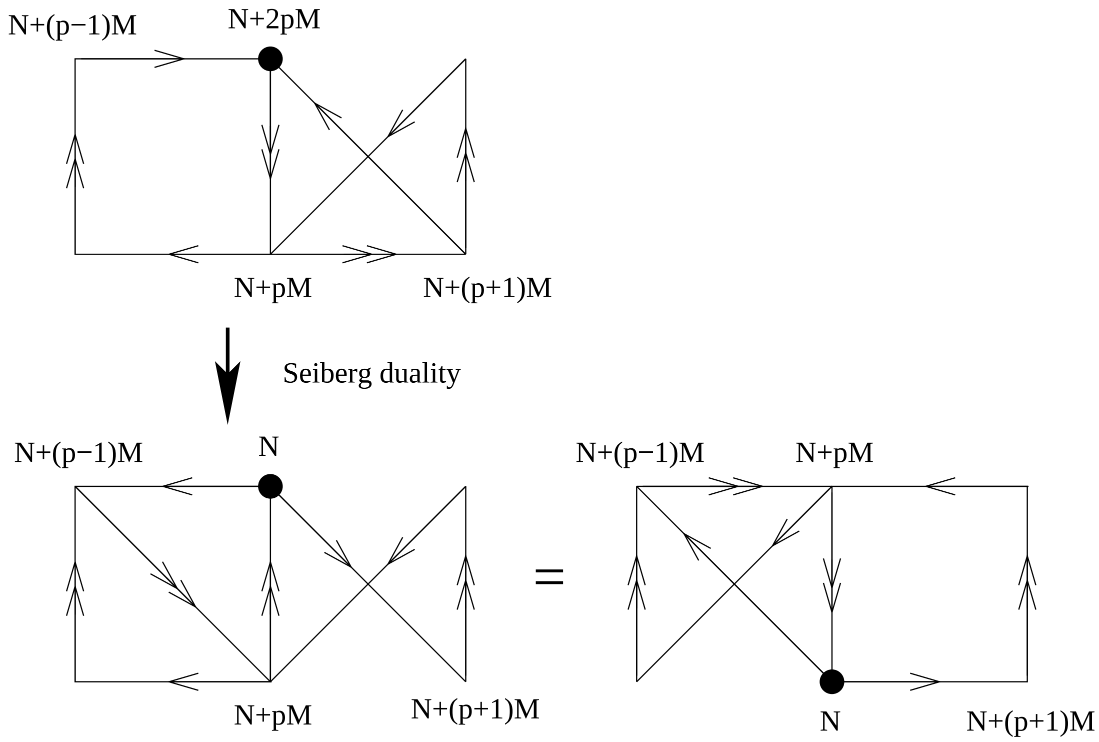

Figure 4: Seiberg duality for the $Y^{p,1}$ quiver: $(\cdots\tau\tilde{\tau}\sigma\tilde{\tau}\cdots) \to (\cdots\tau\tilde{\sigma}\tau\tilde{\tau}\cdots)$.

7.3 The baryonic branch

Both for $Y^{p,1}$ and $Y^{p,p-1}$, if initially N is a multiple of M then far in the IR N is reduced to zero, so that we find the gauge group $\mathrm{SU}(M)\times\mathrm{SU}(2M)\times\ldots\times\mathrm{SU}(2pM)$. Note that for the $\mathrm{SU}(2pM)$ factor there are effectively $2pM$ flavors. Hence we can form baryon operators. In this sense the cascade obtained is rather analogous to the cascade found with $T^{1,1}$ (the latter case formally corresponds to $p=1$ and $q=0$). It is therefore possible that all these theories have a baryonic branch where the $\mathrm{U}(1)_B$ and the $\mathrm{U}(1)_F$ continuous symmetries are spontaneously broken. This idea needs further investigation because the dynamics of the $\mathrm{SU}(M)\times\mathrm{SU}(2M)\times\ldots\times\mathrm{SU}(2pM)$ gauge theory is necessarily more complex than for the $\mathrm{SU}(M)\times\mathrm{SU}(2M)$ case found for the deformed conifold.

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